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WILD EDGE COLOURINGS OF GRAPHS

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Abstract. We prove consistent, assuming there is a supercompact cardinal, that there is a singular strong limit cardinal μ, of cofinality ω, such that every μ^+-chromatic graph X on μ^+ has an edge colouring c of X into μ colours for which every vertex colouring g of X into at most μ many colours has a g-colour class on which c takes every value.

The paper also contains some generalisations of the above statement in which μ^+ is replaced by other cardinals $> \mu$.

§1. Introduction and the statement of the Theorem. In [4] Hajnal and Komjáth study the question of the existence of an edge colouring of a graph with the property that whenever one takes a vertex colouring of the graph there is a colour class where the edge colouring assumes all colours. A typical situation is when the graph X is of chromatic number $\chi(X) \geq \mu^+$ and one seeks a c, a μ^+-colouring of the edge set $E(X)$ of X, such that for every μ-colouring $g : X \to \mu$ of the vertex set of X there is some colour class $g^{-1}\{i\}$ on which c assumes every value. Note that X must be at least μ^+-chromatic for such a c to exist. They denote the statement, that for every graph X with chromatic number $\chi(X) = \mu^+$ one can find such an edge set μ^+-colouring c by $Q(\mu^+)$.

The motivation for the study of this concept comes from the investigation of obligatory finite configurations in triple systems, and this and some of the earlier history of the question can be found in [4]. An interesting point to note is that the question of whether $Q(\aleph_1)$ is true in *ZFC* is still open. The particular instance of $Q(\aleph_1)$ where the graph X is the complete graph on $\aleph_1$ vertices was answered in the affirmative by Todorčević in [7], and even though this is a just one specific case of the general question the solution is very involved.

Hajnal and Komjáth study $Q(\mu^+)$ for regular cardinals μ in [4]. They give a variety of results, for example, that if $\mu = 2^{<\mu}$ and a Cohen subset is added to μ using conditions of size $< \mu$ (that is, using $\mathrm{Fn}(\mu, 2, {<}\mu)$ as the forcing notion), then $Q(\mu^+)$ holds in the resulting universe, and that $Q(\aleph_1)$ holds if $\diamondsuit^+$ is true. However, the one most relevant to the theorem presented here is [4, Theorem 2], in which they

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prove that if $\kappa > \mu$ and more than κ subsets of μ are added using Cohen forcing (that is, with $\mathrm{Fn}(\lambda, 2,\mu)$ for some $\lambda > \kappa$), then in the resulting model every graph X of cardinality κ with chromatic number greater than μ admits an edge colouring c into μ colours such that for every vertex colouring into $< \chi(X)$ colours there is a colour class on which c assumes every value.

We start by asking what one can say about $Q(\mu^+)$ for a singular cardinal μ. One cannot hope to generalise the first result of [4] mentioned above naively because, for one thing, we do not have a notion of adding a Cohen subset to a singular cardinal. In fact, the combinatorial structure of successors of singular cardinals is a subject entirely different from that of the successors of regular cardinals, because some of this structure depends on the situation at the regular cardinals below. This very fascinating phenomenon, which we have described in only the roughest of terms, has been a subject of study in set theory for almost forty years, and many of the difficult open questions in set theory can be traced back to it in one way or another. One can consult the introduction of Shelah's book [6] for a survey. It is, in particular, true that even the simplest consistency results at the successor of a singular cardinal have large cardinal consistency strength.

Another approach to forcing statements, like $Q(\mu^+)$, with a leading universal quantifier when μ is regular is to use an iteration of forcings, in some way dealing with each possible instantiation of the quantifier at some step in the iteration. However, techniques for iterating at the successor of a singular cardinal are difficult and, in fact, very little is known about the subject at all as yet.

One of the few available methods for forcing the consistency of statements with a leading universal quantifier at the successor of a singular, is presented in a paper by Džamonja and Shelah [1], where they gave an axiomatic approach to the question. They showed that under favourable circumstances, a certain kind of forcing can be "iterated" at the successor of a singular cardinal of cofinality ω. The quotation marks are to indicate that this "iteration" is not an analogue of what one sees at, say, $\aleph_1$. Rather, one starts with a large cardinal μ, does an iteration of forcing at μ making sure that μ remains measurable, and then follows this by a Prikry forcing to change the cofinality of μ to ω. The point is to assure during the iteration that the situation after the iteration *and* the Prikry forcing is well controlled during the iteration itself. Here we shall apply the axiomatic framework of [1] to prove our main result. The individual steps of our iteration are based on Hajnal and Komjáth' proof of [4, Theorem 2].

THEOREM. *Assume that V is a universe in which μ is a supercompact cardinal. Then for every pair $\tau \leq \kappa$ of cardinals with $\mu < \tau$ there is a generic extension of V in which μ is singular strong limit of cofinality ω, cardinals and cofinalities $\geq \mu^+$ and $< \mu$ are preserved (so $\mu < \tau \leq \kappa$ in the extension), while every graph $\mathbf{G}$ of size at most κ and chromatic number τ admits an edge colouring $c\colon E(\mathbf{G}) \to \mu$ such that for every vertex colouring $g\colon \mathbf{G} \to \phi$ with $\phi < \tau$ there is $i < \phi$ such that $c\restriction(E(\mathbf{G}) \cap [g^{-1}``\{i\}]^2)$ assumes all colours.*

In particular, if $\tau = \mu^+$, then we obtain a statement close to $Q(\mu^+)$. The main difference between the statement we obtain and $Q(\mu^+)$ is that we need to require our graphs to have size $\leq \kappa$ (in fact our result is closer to the statement of [4, Theorem 2].

§**2. Proof of the Theorem.** In this section we give the proof of the Theorem.

PROOF. We start with a universe V in which μ is a supercompact cardinal and τ and κ are as in the statement of the Theorem. By the well known results of Laver [5] we may assume that the following are true of V:

1. (1) GCH holds at and above μ,
2. (2) the supercompactness of μ is indestructible under μ-directed closed forcing (i.e., forcing in which every system of fewer than μ many elements which are pairwise compatible within the system has a common extension).

NOTATION 2.1.

(1) If $\mathscr{D}$ is a measure on μ then $\Pr(\mathscr{D})$ is Prikry forcing at μ using $\mathscr{D}$ as the ultrafilter: the forcing in which conditions are of the form (s, S) with $s \in [\mu]^{<\omega}$, $S \in \mathscr{D}$ and $\max(s) < \min(S)$ and with (s^*, S^*) refining (s, S) iff s is an initial segment of s^* and $(s^* \setminus s) \cup S^* \subseteq S$.
(2) If E is a binary relation giving a graph on an ordinal α, when we write $E\{\delta, \varepsilon\}$ or $E(\delta, \varepsilon)$ we assume $\delta < \varepsilon$.
(3) For a set A of ordinals we let $\operatorname{ssup}(A)$ be the least α such that $A \subseteq \alpha$.
(4) If s and w are finite sequences (of ordinals) then $s^\frown w$ is the concatenation of s and w.

As we mentioned in the introduction, our forcing extension will be obtained by an iteration of cardinal-preserving forcings. After an initial forcing which renders $2^\mu \geq \kappa$ (by adding Cohen subsets of μ), in each intermediate universe μ will still be a measurable cardinal and we shall be given a cardinal $\lambda \in [\tau, \kappa]$ and a measure $\mathscr{D}$ on μ, both depending on the universe.

DEFINITION 2.2. Let $V' \supseteq V$ be any universe in which $\tau \leq \lambda \leq \kappa \leq 2^\mu$ and $\mathscr{D}$ is a measure on μ (in particular V' might be an intermediate extension in our forcing). Suppose we are given a canonical (in the usual sense) $\Pr(\mathscr{D})$-name $(\lambda, \dot{E})$ for a graph on λ. Further suppose that $(s, S) \in \Pr(\mathscr{D})$ is a condition and $\dot{e}$ is a name such that

$$(s, S) \Vdash_{\Pr(\mathscr{D})} \text{“}\dot{e}\colon \lambda \to \dot{E} \text{ is a bijection } \& \chi((\lambda, \dot{E})) = \tau\text{”}.$$

We define a forcing notion $Q \stackrel{\text{def}}{=} Q_{\mathscr{D}}^{V'}(\dot{E}, (s, S))$ as follows.
If $p = \langle A^p, B^p, t^p, f^p \rangle$, then $p \in Q$ if:

(i) $A^p \in [S]^{<\mu}$,
(ii) $B^p \in \mathscr{D}$, $B^p \subseteq S$ and $\operatorname{ssup}(A^p) < \min(B^p)$,
(iii) $t^p \in [\lambda]^{<\mu}$,
(iv) f^p is a function from t^p into μ,
(v) for all $w \in [A^p]^{<\omega}$ and for all $\alpha \in t^p$, there are $\delta, \varepsilon < \lambda$ such that

$$(s^\frown w, B^p) \Vdash_{\Pr(\mathscr{D})} \text{“}\dot{e}(\alpha) = \{\delta, \varepsilon\}\text{”}.$$

We define the order on Q by letting q refine p if and only if

(iv) $A^q \supseteq A^p$, $B^q \subseteq B^p$, $f^q \restriction t^p = f^p$ and $A^q \setminus A^p \subseteq B^p$.

PROPOSITION 2.3.

(1) *Suppose that G is Q-generic over V', and set $A^* \stackrel{\text{def}}{=} \bigcup\{ A^p \mid p \in G \}$ in $V'[G]$. Then $|A^*| = \mu$ and $|A^* \setminus B| < \mu$ for every $B \in \mathscr{D}$.*

(2) *The set* $\mathscr{K}_\alpha \overset{\text{def}}{=} \{ p \in Q \mid \alpha \in t^p \}$ *is dense and open in* Q *for each* $\alpha < \lambda$.
(3) Q *is separative.*
(4) Q *is* μ*-directed-closed.*
(5) *The set* $\mathscr{J}_\gamma \overset{\text{def}}{=} \{ p \in Q \mid \gamma \in \operatorname{rge}(f^p) \}$ *is dense and open in* Q *for each* $\gamma < \mu$.
(6) *The set* $\{ f^p \mid p \in Q \}$ *is* $\operatorname{Fn}(\lambda, \mu, {<}\mu)$, *i.e., the set of functions making up the forcing to add* λ *many Cohen subsets of* μ *by conditions of size* $< \mu$.

Proof.
(1) We notice that $\mathscr{I}_i \overset{\text{def}}{=} \{ p \in Q \mid \exists j \geq i \ (j \in A^p) \}$ is dense in Q for each $i < \mu$. For if $p \in Q$ and $p \notin \mathscr{I}_i$ let $j \in B^p \setminus i$ and define $q \in \mathscr{I}_i$, refining p, by setting $A^q \overset{\text{def}}{=} A^p \cup \{j\}$, $B^q \overset{\text{def}}{=} B^p \setminus (j+1)$, $t^q \overset{\text{def}}{=} t^p$ and $f^q \overset{\text{def}}{=} f^p$. q satisfies (v) in the definition of Q because if $w \in [A^q]^{<\omega}$ we have that $(s^\frown w, B^q)$ refines $(s^\frown w \setminus \{j\}, B^p)$ in $\operatorname{Pr}(\mathscr{D})$; hence q is an element of Q. So $|A^*| = \mu$.

For $B \in \mathscr{D}$ the set $\mathscr{L}_B \overset{\text{def}}{=} \{ p \in Q \mid B^p \subseteq B \}$ is dense and open in Q. If $p \in \mathscr{L}_B \cap G$, then for any $q \in G$ that extends p we have $A^q \setminus A^p \subseteq B^p$. Hence $A^* \setminus B \subseteq A^p$.

(2) Let $\alpha < \mu^+$ and $p \in Q$. If $p \in \mathscr{K}_\alpha$ then there is nothing more to do. So suppose that $p \notin \mathscr{K}_\alpha$, i.e., $\alpha \notin t^p$. Let $\gamma \in \mu \setminus \operatorname{rge}(f^p)$. We have that $(s, S) \Vdash_{\operatorname{Pr}(\mathscr{D})}$ "$\exists \{\delta, \varepsilon\} \in [\lambda]^2 \ (\dot{e}(\alpha) = \{\delta, \varepsilon\})$", so applying the Prikry Lemma, for every $w \in [A^p]^{<\omega}$ there is some $B_w \subseteq B^p$ with $B_w \in \mathscr{D}$ and some $\{\delta_w, \varepsilon_w\} \in [\lambda]^2$ such that $(s^\frown w, B_w) \Vdash_{\operatorname{Pr}(\mathscr{D})}$ "$\dot{e}(\alpha) = \{\delta_w, \varepsilon_w\}$".

We define q by letting $A^q \overset{\text{def}}{=} A^p$, $B^q \overset{\text{def}}{=} \bigcap \{ B_w \mid w \in [A]^{<\omega} \}$, $t^q \overset{\text{def}}{=} t^p \cup \{\alpha\}$ and $f^q \overset{\text{def}}{=} f^p \cup \{(\alpha, \gamma)\}$.

(3) This is proved using an argument very similar to the one for (2). As observed there, given a condition p we can easily find a condition in Q refining it with a non-empty t-part, so without loss of generality we may as well assume $t^p \neq \emptyset$. Let $\alpha \in t^p$. Working exactly as before we can define two extensions q_0, q_1 of p which are both in $\mathscr{K}_\alpha$ but satisfy $f^{q_0}(\alpha) \neq f^{q_1}(\alpha)$, simply by choosing, at the relevant stage of that argument, two distinct elements γ_0 and γ_1 of $\mu \setminus \operatorname{Rang}(f^p)$, where previously we merely picked one. Then q_0 and q_1 are incompatible.

(4) Suppose that $i^* < \mu$, that $p_i \overset{\text{def}}{=} \langle A^i, B^i, t^i, f^i \rangle$ for each $i < i^*$ and that $\{ p_i \mid i < i^* \}$ is directed. We define $A \overset{\text{def}}{=} \bigcup_{i<i^*} A^i$, $B \overset{\text{def}}{=} \bigcap_{i<i^*} B^i$, $t \overset{\text{def}}{=} \cup_{i<i^*} t^i$, and for $\alpha \in t$ we let $f(\alpha) \overset{\text{def}}{=} f^i(\alpha)$ for any $i < i^*$ such that $\alpha \in t^i$. It is easy to see that $\langle A, B, t, f \rangle$ is in Q and refines each p_i.

(5) Similar to (2).

(6) Suppose $f \in \operatorname{Fn}(\lambda, \mu, {<}\mu)$ with $\operatorname{dom}(f) = t$. Define a condition $p \overset{\text{def}}{=} \langle \emptyset, B^p, t, f \rangle \in Q$ by iterating the application of the Prikry Lemma in (2) through the elements of t in order to ensure p satisfies (v) in the definition of Q. $\dashv$

Proposition 2.4. *Q is μ-centred* (*in other words, Q can be written as a union of μ sets each of which consists of pairwise compatible conditions*).

Proof. Let $\{ p_i \mid i < \theta \}$ be some 1-1 enumeration of Q, where $\theta = |Q| = \lambda^{<\mu}$, and for each $i < \theta$ suppose that $p_i = \langle A^i, B^i, t^i, f^i \rangle$. Note that there are sets $\{ A^*_\zeta \mid \zeta < \mu \}$ such that for each $i < \theta$ there is some $\zeta < \mu$ such that $A^i = A^*_\zeta$. It clearly suffices to show that for every $\zeta < \mu$ the set $\{ p_i \mid A_i = A^*_\zeta \}$ is μ-centred, so we assume without loss of generality that we are instead given an enumeration of a subset of Q satisfying that there is a set A^* such that for all $i < \theta$ we have $A^i = A^*$. Similarly (since $\lambda \leq 2^\mu$), the Cohen forcing to add λ many Cohen sets to μ is μ-centred, so we can assume that for all $i, j < \lambda$ the conditions f^i and f^j are compatible in that forcing. Now we claim that we are left with a set of pairwise compatible conditions. If $i, j < \theta$ define q refining p^i and p^j by

$$q \stackrel{\text{def}}{=} \langle A^*, B^i \cap B^j, t^i \cup t^j, f^i \cup f^j \rangle.$$

Note that for any $\alpha \in t^i \cup t^j$, there is $k \in \{i, j\}$ such that $\alpha \in t^k$. Hence for every $w \in [A^*]^{<\mu}$ we have that

$$(s^\frown w, B^i \cap B^j) \Vdash_{\Pr(\mathscr{D})} \text{“}\dot{e}(\alpha) = \{\delta, \varepsilon\}\text{”}$$

for some $\{\delta, \varepsilon\} \in [\lambda]^2$, and so $q \in Q$. ⊣

Lemma 2.5. *Let V', $\mathscr{D}$ be as defined in Definition 2.2, in particular $\mathscr{D} \in V'$ is a normal ultrafilter over μ. Suppose that P is a forcing notion in V' such that*

$$\Vdash_P \text{“}\mathscr{D} \subseteq \dot{\mathscr{D}}' \textit{ and } \dot{\mathscr{D}}' \textit{ is a normal ultrafilter over } \mu\text{”}.$$

Then $i\colon \Pr(\mathscr{D}) \lhd P * \Pr(\dot{\mathscr{D}}')$ given by $i((a, A)) \stackrel{\text{def}}{=} (\mathbf{1}_P, (a, A))$ is a complete embedding.

Proof of the Lemma. Routine checking. ⊣

The embedding i extends in a natural way, through recursion on the definition of $\Pr(\mathscr{D})$-names, to an embedding of $\Pr(\mathscr{D})$-names into $P * \Pr(\dot{\mathscr{D}}')$-names. Through this embedding we have that $\Vdash_{\Pr(\mathscr{D})}$ “$\{\delta, \varepsilon\} \in \dot{E}$” if and only if $\Vdash_{P*\Pr(\dot{D}')}$ “$\{\delta, \varepsilon\} \in i(\dot{E})$” for each $\{\delta, \varepsilon\} \in [\lambda]^2$.

So for (s, S) and $\dot{e}$ as in as in Definition 2.2 we also have that

$$(\mathbf{1}_P, (s, S)) \Vdash_{P*\Pr(\dot{D}')} \text{“}i(\dot{e})\colon \lambda \longrightarrow \dot{E} \text{ is a bijection”}$$

and

$$(s, S) \Vdash_{\Pr(\mathscr{D})} \text{“}\{\delta, \varepsilon\} \in \dot{E} \text{ and } \dot{e}(\beta) = \{\delta, \varepsilon\}\text{”}$$

if and only if

$$(\mathbf{1}_P, (s, S)) \Vdash_{P*\Pr(\dot{\mathscr{D}}')} \text{“}\{\delta, \varepsilon\} \in i(\dot{E}) \text{ and } i(\dot{e})(\beta) = \{\delta, \varepsilon\}\text{”}.$$

Definition 2.6. Suppose that $Q \lhd P$ via a function $j\colon Q \longrightarrow P$, and $\dot{\mathscr{D}}'$ is a P-name of a normal ultrafilter over μ with $\dot{\mathscr{D}}' \supseteq \mathscr{D} \cup \{\dot{A}^*\}$. Let $\dot{G}$ be a canonical $P * \Pr(\dot{\mathscr{D}}')$-name for a $P * \Pr(\dot{\mathscr{D}}')$-generic filter. We define a $P * \Pr(\dot{\mathscr{D}}')$-name $\dot{c} = \dot{c}^{\dot{\mathscr{D}}'}$ for a colouring of $[\lambda]^2$ with μ colours by setting for each pair $\{\delta, \varepsilon\}$

$\dot{c}(\{\delta, \varepsilon\}) = \gamma$ if and only if

$\exists (j(p), (s^\frown w, B^p)) \in \dot{G}$ with $p \in Q$, $w \in [A^p]^{<\omega}$ and $B^p \subseteq S$ such that for some $\alpha \in t^p$ we have $f^p(\alpha) = \gamma$ and $(s^\frown w, B^p) \Vdash_{\Pr(\mathscr{D})}$ “$\dot{e}(\alpha) = \{\delta, \varepsilon\}$”.

Claim 2.7. *Suppose that P is a μ-directed closed forcing, $Q \lhd P$, and that $i\colon \Pr(\mathscr{D}) \lhd P * \Pr(\dot{\mathscr{D}}')$ is as in Lemma 2.5, where $\dot{\mathscr{D}}'$ is a P-name of a normal ultrafilter over μ with $\dot{\mathscr{D}}' \supseteq \mathscr{D} \cup \{\dot{A}^*\}$. Then*

$$(\mathbf{1}_P, (s, \dot{A}^*)) \Vdash_{P*\Pr(\dot{\mathscr{D}}')} \text{“}\dot{c} \text{ is an edge colouring of } i(\dot{E}) \text{ whose range is } \mu\text{”}$$

Proof of the Claim. Let G be a $P * \Pr(\dot{\mathscr{D}}')$-generic over V' such that $(\mathbf{1}_P, (s, \dot{A}^*)) \in G$ and suppose that in $V'[G]$ it is not true that c is a colouring of E the range of which is μ. There must be some $p^+ \in G$ such that $p^+ \Vdash$ "$\dot{c}$ is not a function from $i(\dot{E})$ onto μ", and hence there must be some q^+ refining p^+ with $q^+ \in G$ and some ordinals $\delta,\ \varepsilon < \lambda, \gamma < \mu$ such that either $q^+ \Vdash$ "$i(\dot{E})\{\delta,\varepsilon\}$ & $\neg\{\delta,\varepsilon\} \in \mathrm{Dom}(\dot{c})$" or $q^+ \Vdash$ "$\gamma \notin \mathrm{rge}(\dot{c})$". Let us identify P with $Q * \dot{P}/Q$ and set $q^+ = (p, q, r)$, where the factoring corresponds to $Q * \dot{P}/Q * \Pr(\dot{\mathscr{D}}')$, while $r = (w, \dot{B})$. Note that $w \in V'$ is an object not a name.

Firstly, suppose $q^+ \Vdash$ "$i(\dot{E})\{\delta,\varepsilon\}$ & $\neg\{\delta,\varepsilon\} \in \mathrm{Dom}(\dot{c})$". Since both $(\mathbf{1}_P, (s, \dot{A}^*))$ and q^+ are elements of G, by extending and renaming if we need to, we may as well assume that q^+ extends $(\mathbf{1}_P, (s, \dot{A}^*))$. By extending q^+ and renaming again if necessary, we can assume that for some ordinal $\beta < \lambda$ the condition q^+ forces that $i(\dot{e})(\beta) = \{\delta,\varepsilon\}$. By extending q^+ again if necessary, we may assume that $\beta \in t^p$. In particular, it is forced by (p, q) that $w \in [\dot{A}^*]^{<\omega}$. As $\dot{A}^*$ is a Q-name and $w \in V'$, it is already forced by $p \in Q$ that $w \in [\dot{A}^*]^{<\omega}$. By extending and renaming again if necessary, we may assume that $w \subseteq A^p$. By the requirement (v) of the definition of Q, there are $\delta', \varepsilon' < \lambda$ such that

$$(s^\frown w, B^p) \Vdash_{\Pr(\mathscr{D})} \text{“}\dot{e}(\beta) = \{\delta', \varepsilon'\}\text{”}.$$

Since P forces that $\mathscr{D} \cup \{\dot{A}\}^* \subseteq \dot{\mathscr{D}}'$, the conditions $(s^\frown w, \dot{B})$ and $(s^\frown w, B^p)$ are forced to be compatible in $\Pr(\dot{\mathscr{D}}')$, and since $i(\dot{e})$ is forced to be a bijection, we conclude that $\delta' = \delta$ and $\varepsilon' = \varepsilon$. We may assume, by extending q^+ again if need be that (p, q) forces that $\dot{B} \subseteq B^p$. Then q^+ is an extension of the condition $r^+ \overset{\text{def}}{=} (p, \mathbf{1}_{P/Q}, (s^\frown w, B^p))$, so $r^+ \in G$. By the definition of $\dot{c}$ we have that $\dot{c}(\{\delta,\varepsilon\})$ is forced to be $f^p(\beta)$, so in particular $\{\delta,\varepsilon\} \in \mathrm{Dom}(c)$ holds in $V'[G]$, contrary to our assumptions.

So suppose instead that the second alternative holds, so $q^+ \Vdash$ "$\gamma \notin \mathrm{rge}(\dot{c})$" for some $\gamma < \mu$. By Proposition 2.3(5), we can assume that for some $\alpha \in t^p$ we have $f^p(\alpha) = \gamma$, that (p, q) forces that $\dot{B} \subseteq B^p$ and that p forces that $w \in [\dot{A}^*]^{<\omega}$. Condition (v) in the definition of Q allows us to find $\{\delta,\varepsilon\}$ such that $(s^\frown w, B^p) \Vdash_{\Pr(\mathscr{D})}$ "$\dot{e}(\alpha) = \{\delta,\varepsilon\}$", and then arguing as before we conclude that $c(\{\delta,\varepsilon\}) = \gamma$ holds in $V'[G]$, a contradiction. This finishes the proof. ⊣

The next Claim will use Fodor's set mapping theorem, which we now recall.

Definition 2.8.

(1) The mapping $F\colon X \to \mathscr{P}(X)$ on the set X is said to be a *set mapping* if $x \notin F(x)$ for any $x \in X$.
(2) A set $T \subseteq X$ is *free* with respect to the set mapping $F\colon X \to \mathscr{P}(X)$ if and only if $x \notin F(y)$ for any $x,\ y \in T$.

Theorem 2.9 (G. Fodor [2]. Also consult the textbook [3] for a proof). *If θ is an infinite cardinal and $F : X \to \mathscr{P}(X)$ is a set mapping on X such that $|F(x)| < \theta$ for all $x \in X$, then X can be represented as the union of at most θ many sets each of which is free with respect to F.*

Claim 2.10. *Suppose that Q, P, $\mathscr{D}$ and $\dot{\mathscr{D}}'$ are as in Claim 2.7. Let G be P-generic over V'.*

Suppose also that H is a $\mathrm{Pr}(\dot{\mathscr{D}}')$-generic filter over $V'[G]$ with $(s, \dot{A}^*) \in H$. Let J be a $\mathrm{Pr}(\mathscr{D})$-generic filter over V' induced by $G * H$ (whose existence follows from Lemma 2.5). Let $\mathscr{G} = (\lambda, E) \in V'[J]$ be the τ-chromatic graph such that $E = \dot{E}_J$, and let $c \in V'[G][H]$ be the realisation of $\dot{c}$ defined by Definition 2.6.

Then for every vertex colouring $k\colon \lambda \to \phi$ of $\mathscr{G}$ in $V'[J]$ with $\phi < \tau$ there is some $\gamma < \phi$ such that $c``([k^{-1}(\{\gamma\})]^2) = \mu$.

Proof of the Claim. Let G^* be a $P * \mathrm{Pr}(\dot{\mathscr{D}}')$-generic filter over V' such that $V'[G^*] = V'[G][H]$.

Suppose that the conclusion of the claim is not true, and let $k\colon \lambda \to \phi$ be a vertex colouring of $\mathscr{G}$ in $V'[J]$ exemplifying this. For every $\gamma < \mu$ let $j(\gamma) < \mu$ be such that $j(\gamma) \notin c``([k^{-1}(\{\gamma\})]^2)$. For each $\xi < \lambda$ let p_ξ be a condition in $P * \mathrm{Pr}(\dot{\mathscr{D}}')$ which is an extension of $(\mathbf{1}_P, (s, \dot{A}^*))$, an element of G^*, and such that p_ξ decides $i(\dot{k})(\xi)$ and the value $\dot{l}(\xi) \stackrel{\text{def}}{=} \dot{j}(i(\dot{k})(\xi))$.

For $\xi < \lambda$ write $p_\xi = (q_\xi, r_\xi)$ where $q_\xi \in Q$ and, identifying P with $Q * \dot{P}/Q$, $r_\xi \in \dot{P}/Q * \mathrm{Pr}(\dot{\mathscr{D}}')$. In $V'[G][H]$ define

$$g(\xi) \stackrel{\text{def}}{=} \{\, \zeta \neq \xi \mid \exists \sigma < \lambda\, \exists \alpha \in t^{q_\xi}\ (e(\alpha) = \{\sigma, \zeta\} \text{ or } e(\alpha) = \{\zeta, \sigma\})\,\}$$

for $\xi < \lambda$. (Here $e = i(\dot{e})_{G^*}$.) As $|t^{q_\xi}| < \mu$, we also have that $|g(\xi)| < \mu$. Noticing that $\xi \notin g(\xi)$ for any ξ, we have that g is a set mapping on λ, and so by Fodor's Set Mapping Theorem, Theorem 2.9, we can decompose λ into the union of μ many g-free sets. Using this, we can write $\lambda = \bigcup\{\, W_\gamma \mid \gamma < \max(\{\mu, \phi\})\,\}$ so that if $\xi, \zeta \in W_\gamma$ then the value of $i(\dot{k})(\zeta)$ as decided by p_ζ is the same as the value of $i(\dot{k}(\xi))$ as decided by p_ξ, the values $\dot{l}(\xi)$ and $\dot{l}(\zeta)$ are also the same, p_ξ and p_ζ are compatible, and $\xi \notin g(\zeta)$. Since the chromatic number of E is forced by $(s, S)(\in J)$ to be τ, which is greater than both μ and ϕ, there is $\gamma < \max(\{\mu, \phi\})$ and $\xi \neq \zeta \in W_\gamma$ such that $\{\xi, \zeta\} \in E$ holds in $V'[J]$. Let $\alpha < \lambda$ be such that $e(\alpha) = \{\xi, \zeta\}$ holds in $V'[J]$. By the definition of g, this implies that $\alpha \notin t^{q_\zeta} \cup t^{q_\xi}$. Noticing that any two conditions that are compatible in Q have a common refinement whose t-part is the union of the t-parts of the two conditions in question, we can find $p = (q, r)$ that extends both p_ξ and p_ζ and such that $t^q = t^{q_\xi} \cup t^{q_\zeta}$. Extending p further and renaming, we can assume that $\alpha \in t^q$ and $f^q(\alpha) = l(\xi) (= l(\zeta)) \stackrel{\text{def}}{=} l^*$. But then p forces $\dot{l}^*$ to be $\dot{c}(k^*)$, where k^* is the common value forced to $i(\dot{k})(\zeta), i(\dot{k})(\xi)$ and such that $\dot{j}(k^*) = l^*$, in contradiction to what is forced about $\dot{j}(k^*)$ by either of the weaker conditions p_ξ and p_ζ. This contradiction shows that a bad vertex colouring could not have existed, and finishes the proof. $\dashv$

Claim 2.11. *Let $\mathscr{D}$ be a normal ultrafilter over μ and $A \in \mathscr{D}$ and $(a, A) \in \mathrm{Pr}(\mathscr{D})$. Suppose that G is a $\mathrm{Pr}(\mathscr{D})$-generic filter over V. Then there is some G' which is $\mathrm{Pr}(\mathscr{D})$-generic over V and such that $(a, A) \in G'$ while $V[G] = V[G']$.*

Proof of the Claim. Let $x_G = \cup\{\, w \mid \exists B \in \mathscr{D}\ ((w, B) \in G)\,\}$, and for $x \subseteq \mu$ put

$$G_x = \{\, (w, B) \in \mathrm{Pr}(\mathscr{D}) \mid w \subseteq x \subseteq w \cup B \,\}.$$

Now we use the Mathias characterisation of Prikry forcing, which says that for an infinite subset x of μ we have that G_x is $\mathrm{Pr}(\mathscr{D})$-generic over V if and only if $x \setminus B$ is finite for all $B \in \mathscr{D}$. Hence $x_G \setminus A$ is finite. Let $y = a \cup (x_G \cap A)$. Then y is an infinite subset of μ which clearly satisfies that $y \setminus B$ is finite for all $B \in \mathscr{D}$. Let $G' = G_y$. G' is $\Pr(\mathscr{D})$-generic over V and $(a, A) \in G'$. We have $V[G'] \subseteq V[G]$ because $y \in V[G]$, and $V[G] \subseteq V[G']$ because $x_G \setminus y$ is finite. $\dashv$

Corollary 2.12. *Suppose that Q, P, $\dot{\mathscr{D}}'$, G and V' are as in Claim* 2.10 *and H is a* $\Pr(\mathscr{D}')$*-generic filter over $V'[G]$. Then the conclusion of Claim* 2.10 *holds in $V'[G][H]$ by Claim* 2.11.

The principal part of the remainder of the proof consists of checking that our one step forcing Q satisfies a certain condition taken from [1] (which we rehearse below). It was proved in [1] that when one has Q-like forcings satisfying the condition there is an iteration of them that will build a normal ultrafilter over μ that can be used after the iteration to define a Prikry extension, and, at the same time, do the bookkeeping needed to ensure that all the candidate graphs and conditions in the Prikry forcing are considered, so that ultimately the desired conclusion will hold in the Prikry extension. (Without indulging in a precise definition of Q-like we remark that conditions in such forcings all have the form $p = (A^p, B^p, \dots)$, where A^p, B^p are as in Definition 2.2, (i), (ii).)

To this end we start with the universe V that we defined at the beginning of the proof and a cardinal $\Upsilon \geq \kappa^+$. Using the notation of Definition 1.9 of [1] we define a sequence $\bar{Q} = \langle P_i, \dot{Q}_i, \dot{A}_i : i < \Upsilon^+ \rangle$ in $\mathscr{K}^+_{\Upsilon^+}$ by induction on i, using for Q_0 the Cohen forcing for adding κ many subsets of μ by conditions of size $< \mu$, and, for each $i > 0$, letting $\dot{Q}_i = Q^{V^{P_i}}_{\dot{\mathscr{D}}_i}(\dot{E}_i, (s_i, \dot{S}_i))$ for some $\dot{\mathscr{D}}_i, \dot{E}_i$, and $(s_i, \dot{S}_i)$ as in Definition 2.2 with $V' = V^{P_i}$, and choosing for $\dot{A}_i$ a Q_i-name for A^* of Proposition 2.3(1). When defining this sequence we make sure that the bookkeeping is done so that all relevant names of graphs of size $\leq \kappa$ are mentioned unboundedly often, by the usual methods. We wish to prove that this sequence is fitted, as per Definition 1.20 of [1] (although we will actually show that its components satisfy of a condition sufficient to guarantee fittedness of the sequence rather than fittedness itself, as explained below).

Assuming for the moment that the sequence is fitted, we can extract from $\bar{Q}$, using Main Claim 1.18 of [1], a cardinal and cofinality preserving iteration $\bar{P} = \langle P'_i, \dot{Q}'_j : i \leq \kappa^+, j < \kappa^+ \rangle$ with $<\mu$ supports of forcing in which Q'_0 is the Cohen forcing for adding κ many subsets of μ by conditions of size $< \mu$, and for every $1 \leq i < \kappa^+$ we have that $\dot{Q}'_i = Q^{V^{P'_i}}_{\dot{\mathscr{D}}_i}(\dot{E}_i, (s_i, \dot{S}_i))$ for some $\dot{\mathscr{D}}_i, \dot{E}_i$, and $(s_i, \dot{S}_i)$, and satisfying that in $V^* \stackrel{\text{def}}{=} V^{P'_{\kappa^+}}$ the sequence $\langle \mathscr{D}_i \mid \operatorname{cf}(i) \geq \mu^+ \rangle$ is an increasing sequence of filters on μ, whose union is an ultrafilter $\mathscr{D}^*$ containing all $\{ A^*_i \mid \operatorname{cf}(i) \geq \mu^+ \}$. In addition, the bookkeeping is done so that for every $(s, S) \in \Pr(\mathscr{D}^*)$ and every canonical $\Pr(\mathscr{D}^*)$-name forced by (s, S) to be a τ-chromatic graph $\dot{E}$ on $\lambda \leq \kappa$, there are unboundedly many i of cofinality at least μ^+ such that $\dot{E}$ and $\dot{E}_i$ are the same and $(s_i, \dot{S}_i)$ is the same as $(s, \dot{S})$. After dealing with fittedness and obtaining this iteration we shall show how to finish the proof of the theorem.

We now return to the question of the fittedness of $\bar{Q}$. Definition 1.20 of [1] gives the exact meaning of the word fittedness. However, this rather involved definition will not be necessary for us here because Crucial Claim 1.21 of [1] gives an easy condition on the individual forcings Q which is shown in that claim to be sufficient for the fittedness of the sequence $\bar{Q}$. The condition is exactly that there exists a definition $\dot{d}$ as in the following claim.

Claim 2.13. *There is a definition $\dot{d}$ in V such that whenever $V \subseteq V'$ and $Q = Q^{V'}_{\mathscr{D}}(\dot{E}, (s, S))$ is as defined in Definition 2.2 the value of $\dot{d}_Q$ is a Q-name over V' of a function from Q into $\mathscr{D}$ such that for every regular $\theta < \mu$ and every $\mathbf{g}$ a θ-directed family of conditions in Q of size $< \mu$, such that*

$$r \in \mathbf{g} \Longrightarrow \theta \in \dot{d}_Q(r),$$

there is some $q \in Q$ which is a common refinement of all conditions in $\mathbf{g}$ and such that $q \Vdash$ "$\theta \in \dot{A}$".

Proof of the Claim. Working in V'^Q we define $d = d_Q \colon Q \to \mathscr{D}$ by letting $d(p) \stackrel{\text{def}}{=} B^p$ for $p = \langle A^p, B^p, t^p, f^p \rangle$. We check that this definition is as required. So suppose that $\theta < \mu$ is regular and $\mathbf{g}$ is a θ-directed family of conditions in Q of size $< \mu$ with the property that for all $p \in \mathbf{g}$ we have $\theta \in B^p$. We define r by first applying the directed closedness of Q to $\mathbf{g}$ and then the first part of the proof of 2.3(1) to ensure θ is an element of A^r. ⊣

To finish the proof of the theorem we, as explained above, use Crucial Claim 1.21 of [1] to obtain that $\bar{Q}$ is fitted, and then Main Claim 1.18 of [1] which guarantees that from $\bar{Q}$ we can extract an iteration $\bar{P}$ as required above. We force by $\Pr(\mathscr{D}^*)$ in V^* and claim that the desired conclusion holds in the resulting universe V^+. To check this, suppose that we are given a τ-chromatic graph (λ, E) in V^+ and a vertex colouring $k \colon \lambda \longrightarrow \mu$, and $\lambda \leq \kappa$.

Let $(s, S) \Vdash_{\Pr(\mathscr{D}^*)}$ "the cardinality of $\dot{E}$ is $\lambda \leq \kappa$ & $\chi(\dot{E}) = \tau$," and let π_i be the natural embedding of $\Pr(\mathscr{D}_i)$-names into $\Pr(\mathscr{D}^*)$-names $\Vdash_{\Pr(\mathscr{D}^*)}$ "$\pi_i(\dot{k}_i) = \dot{k}$". Then there is $i < \kappa^+$ with $\mathrm{cf}(i) \geq \mu^+$ such that $\pi_i(\dot{E}_i)$ is a name for (λ, E), $(s_i, \dot{S}_i) \in \Pr(\mathscr{D}_i)$ a name for (s, S) and there is a $\Pr(\mathscr{D}_i)$-name $\dot{k}_i$ for k in $V^{P'_i}$ Also, as all cardinals are preserved by the iteration, $(s_i, \dot{S}_i) \Vdash_{\Pr(\dot{\mathscr{D}}_i)}$ "the cardinality of $\dot{E}_i$ is λ, and $\chi((\lambda, \dot{E}_i)) \geq \tau$".

We pause here to give a justification for the downward preservation of the chromatic number from V^+ to $V^{P'_i}$. Even though the graph already exists in the intermediate universe, it could, a priori, be the case that its chromatic number changes along the iteration and never has the value that it assumes in V^+. This would disqualify the graph from being considered by the bookkeeping. However, the chromatic number of a graph is determined by the existence of a certain function, so it can only decrease from an intermediate universe to the final one. Since an ordinal valued function can decrease only finitely many times, the chromatic number of the graph must eventually stabilise in the iteration, reaching its value from V^+.

Putting this together with the argument from the paragraph prior to the previous one, if i is sufficiently large we have that $(s_i, \dot{S}_i) \Vdash$ "$\chi((\lambda, \dot{E}_i)) = \tau$" (since every function from λ to τ appears before stage κ^+ in the iteration). The conclusion now follows by Corollary 2.12 applied to $Q = Q'_i$, $P = P'_{\kappa^+}$ and $\mathscr{D}' = \mathscr{D}^*$. ⊣

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